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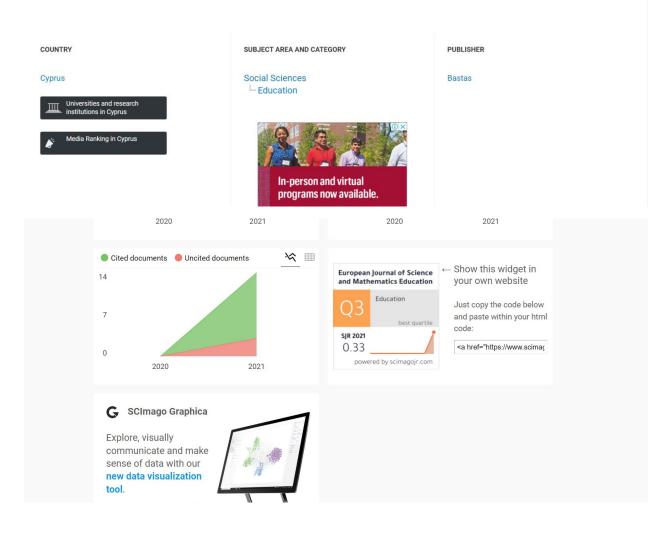
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#### YOUR COMMENTS TO AUTHORS

There are several things that need to be revised in this article:

- Abstract: The abstract section would be better off following the introduction, method, result and conclusion pattern, so that it is clearer (the results section is not yet visible).
- 2. Introduction: Link these two research questions to the results section, so that the coherence is visible 3. Method:
- a. For methodology, it would be better to choose one of several existing types of qualitative methodology in accordance with the development of qualitative methodology, for example grounded theory, case studies, and others. So it is clear theoretically the methodology used
- b. on the observation section, it is necessary to explain what was observed? what do students study? how long is the learning process? what are the materials?
- c. In the interview section, it is necessary to describe what data will be obtained from the interview to be conducted, how many times will the interview process take place? How long? how many respondents were interviewed?
- d. Data analysis steps are not explained
- 4. Result:
- a. In the results section, it would be better if the coding process of the criteria analyzed is added, so that

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- c. In the interview section, it is necessary to describe what data will be obtained from the interview to be conducted, how many times will the interview process take place? How long? how many respondents were interviewed?
- d. Data analysis steps are not explained

4. Result:

- a. In the results section, it would be better if the coding process of the criteria analyzed is added, so that the direction of the research objectives is clear and the reader can immediately identify the findings.
  b. the research results section must be related to the research questions that have been submitted
  c. in the interview section, it would be better to display the results of student work. Thus, the triangulation process will appear more clearly
- 5. Discussions: Before concluding, it would be better to add a discussion because this section is very important to compare with previous findings

#### YOUR COMMENTS FOR EDITORIAL STAFF

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ABSTRACT

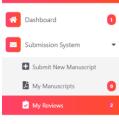
This article needs a major revision because there are several important elements that have not been seen including:

1. Abstract: The abstract section would be better off following the introduction, method, result and conclusion pattern, so that it is clearer (the results section is not vet visible).

The impact of teachers' knowledge on the connection between

technology supported exploration and mathematical proof

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EDITORIAL

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#### YOUR COMMENTS TO AUTHORS

There are several things that need to be revised in this article:

#### Abstrack

 Adjust to the existing methodology in the "Method" section, namely case studies
 The findings in the abstract section should be adjusted to the research questions and the results of answering the research questions

#### Methodology

3. In the methodology section, especially the research stages, the first step is written. But there is no description for the next step (second, third and so on)



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3. In the methodology section, especially the research stages, the first step is written. But there is no description for the next step (second, third and so on)

#### Conclusion

4. The "Conclusion" section may be more suitable for the discussion section because it confirms the findings with the results of previous studies. Next, a conclusion section is made that contains answers to the research questions

#### YOUR COMMENTS FOR EDITORIAL STAFF

1	Digital technology integration and mathematical proof in
2	exploration tasks: the impact of teachers' knowledge
3	ABSTRACT
4	Technology is recognized for its potential to carry out work of an investigative or exploratory
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12	emphasize the relevance of the teacher's MTK - Mathematics and Technology Knowledge, to
13	discuss with the students the conditions to consider when formulating a conjecture and the role
14	of proof; and also the relevance of the teacher's TLTK - Teaching and Learning and
15	Technology Knowledge, to anticipate the students difficulties and support them. The study
16	provides evidence about the difficulty of articulating proof and technology, but it also offers
17	evidence of the relevance of this articulation and of how the teacher's professional knowledge
18	can impact the teacher's options.
19	Keywords: professional knowledge, KTMT, technology, proof
20	[Click here to download the Word file]
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**Commented [L1]:** The abstract section would be better off following the introduction, method, result and conclusion pattern, so that it is clearer (the results section is not yet visible).

## Digital technology integration and mathematical proof in exploration tasks: the impact of teachers' knowledge

#### Abstract:

 Technology is recognized for its potential to carry out work of an investigative or exploratory nature. The ease and speed with which it becomes possible to observe many cases of a given situation, allows the development of conjectures and brings conviction about their veracity. Mathematical proof, assumed as the essence of Mathematics, thus tends to appear to the students as something dispensable. Based on KTMT – *Knowledge for Teaching Mathematics with Technology* model, this study intends to understand the impact of the teachers' knowledge on mathematical proof in a context of technology integration. The study adopts a qualitative and interpretative methodology analyzing the practice of one teacher. The main conclusions emphasize the relevance of the teacher's MTK – *Mathematics and Technology Knowledge*, to discuss with the students the conditions to consider when formulating a conjecture and the role of proof; and also the relevance of the teacher's TLTK – *Teaching and Learning and Technology Knowledge*, to anticipate the students difficulties and support them. The study provides evidence about the difficulty of articulating proof and technology, but it also offers evidence of the relevance of this articulation and of how the teacher's professional knowledge can impact the teacher's options.

Keywords: professional knowledge, KTMT, technology, proof.

#### Introduction

The technology is recognized for its potential for teaching and learning mathematics (Tabach & Trgalová, 2019). In particular, the possibilities of carrying out work of an investigative or exploratory nature are highly valued. It turns possible for the teachers to offer to the students the opportunity to experiment with different mathematical relationships, reflecting on them while trying to identify regularities and formulate conjectures. However, this possibility challenges the teachers' professional knowledge (Rocha, 2020). The ease and speed with which it becomes possible to observe many cases of a given situation, brings conviction about the veracity of the formulated conjecture and fosters a feeling that nothing else is needed to be sure of it (Hsieh et al., 2012; Rocha, 2020). Mathematical proof, assumed as the essence of Mathematics by several authors (Blanton & Stylianou, 2014; Dawkins & Weber, 2017; Rocha, 2019; Schoenfeld, 2009), thus tends to appear to the students as something dispensable (Hanna, 2001).

55 The potential of technology is also related to the ease of access to different representations (Rocha, 56 2016). And, once again, this potentiality challenges the teachers' knowledge. The accessibility and

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57 apparent simplicity of the graphical representation turns the algebraic approach into something that 58 can be circumvented and whose need becomes possible to question. The mastery of algebraic 59 calculations, which in an approach without technology was often the only possible option, thus 60 becomes something expendable. It becomes possible to question the interest in learning and teaching 61 certain algebraic manipulations, as well as the level of fluidity and training that should be required 62 from students.

63 Mathematical proof tends to be related to algebraic approaches (although it does not have to be, as 64 stated by Komatsu (2010)) and the use of technology tends to be related to more intuitive and 65 exploratory approaches based often in graphical representation. As so, not much is known about how 66 to articulate these two approaches. In a previous work (Author), we tried to understand how the 67 teachers conceive proof and an algebraic approach in a context of technology integration, and how they 68 try to turn the algebraic approach relevant to the students. Here, our goal is to understand the impact 69 of the teachers' knowledge on mathematical proof in a context of technology integration. We adopt the 70 KTMT (Knowledge for Teaching Mathematics with Technology) model (Rocha, 2020), giving a special 71 attention to the MTK (Mathematics and Technology Knowledge) and to the TLTK (Teaching and 72 Learning and Technology Knowledge) - two of the main knowledge domains in the KTMT model, as 73 we will see in the next section. Based in this conceptualization and considering the use of exploration 74 tasks<sup>1</sup> at the study of functions in 10<sup>th</sup> grade, we intend to answer the following research questions:

- What is the impact of the teachers' TLTK in mathematical proof while implementing explorations
   in a context of technology integration?
- How does the teachers' MTK influences the options related to mathematical proof while
   implementing explorations in a context of technology integration?

**Commented [L2]:** Link these two research questions to the results section, so that the coherence is visible

#### 80 Mathematical proof

79

81 The literature about mathematical proof has devoted attention to several topics, some of them focusing 82 on the students and some others focusing on the teachers. In what concerns teachers, the research has 83 focused on ways of addressing proof in the classroom and on the teachers' knowledge and professional 84 development (Stylianides, Bieda & Morselli, 2016; Stylianides, Stylianides & Weber, 2017). 85 Nevertheless, and besides all the interest in different topics related to proof and its teaching and 86 learning, not much attention has been given to proof in a context of technology integration.

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87 The understanding about what a mathematical proof is, has changed over time (Smith, 2006), and is 88 not consensual even among mathematicians (Miyakawa, Fujita & Jones, 2017; Steele & Rogers, 2012). 89 Steele and Rogers (2012, p. 161) assume proof as "a mathematical argument that is general to a class of 90 mathematical ideas and establishes the truth of a mathematical statement based on mathematical facts 91 that are accepted or that have been previously proven". Bleiler-Baxter and Pair (2017, p. 16), inspired 92 by De Villiers's (1990) work, define proof as "logical deduction that is used to verify, explain, 93 systematize, discover, and communicate mathematics". In the classroom context, Stylianides and Ball 94 (2008) refer to it as a mathematical argument that uses mathematical knowledge considered valid by 95 the students and that does not require additional justifications, it adopts reasoning considered valid 96 and already known by the students (or whose understanding is within their reach), and which is 97 adequately communicated in ways already familiar to the students (or whose understanding is within 98 their reach).

99 The difficulty in getting students to understand the need for and importance of proof in Mathematics 100 is, according to De Villiers (1999), well known to all secondary school teachers. This difficulty is 101 accentuated when technology is involved because, according to Hsieh et al. (2012), the dynamic 102 character usually offered by it allows the carrying out of work of an experimental nature, which 103 enhances the discovery of properties and the formulation of conjectures. Students can easily experiment 104 and analyze various cases, reflecting on important mathematical ideas and, consequently, reaching a 105 higher level of understanding (Goos & Bennison, 2008). Thus, they acquire the possibility to formulate 106 their own questions and to continue formulating hypotheses and testing them, trying to frame the 107 results in the theory they are trying to formulate (Rocha, 2015).

108 The way in which the analysis of different cases is made possible, ends up giving students a feeling of 109 confidence regarding the veracity of the conclusions they establish with the support of technology, 110 which is often enhanced by the way students got used to seeing Mathematics validated, i.e., externally, 111 either by the teacher, the textbook or even the parents (Tall et al., 2012). The need to prove the 112 formulated conjecture may thus not be felt. But if inferring a conclusion from reflection on some 113 particular cases is an important activity, it is undoubtedly distinct from proving (Cabassut et al., 2012). 114 Emphasizing to the students the need for and importance of proof will then imply the search for its 115 function.

116De Villiers (2012) considers that, traditionally, the justification or convincing about the validity of a117conjecture is seen as the main function of proof, and Knuth (2002) considers that this is really the only118role that most teachers recognize to it. In recent decades, this narrow view of the role of proof has been

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criticized by authors such as Reid (2011), who understand that it has also assumed other important

120 roles for mathematicians and that it can also assume a role of great didactic value in the classroom. 121 For Mejía-Ramos (2005), the search for a deeper understanding is what truly moves mathematicians 122 and what leads them to reject the "alleged" proofs carried out by computational means. A point of view 123 also shared by Bleiler-Baxter and Pair (2017). And this, as highlighted by Hanna (2014), despite the fact 124 of understanding being something remaining relatively undefined. This suggests a role of proof as a 125 means and not so much as an end in itself, encompassing both validation and understanding. In the 126 current reality, in which systems with symbolic algebraic calculus and dynamic geometry programs 127 are easily accessible, it is frequent to obtain a validation of the conjecture with a considerable degree of 128 confidence without a proof (De Villiers, 2012). As so, it becomes difficult to justify the need for a proof 129 exclusively with the need for validation.

119

130 Technologies can convince us of the veracity of the conjecture, but they do not offer us the justification 131 for that veracity (De Villiers, 2012). And this does not seem to be a question exclusive for 132 mathematicians. Indeed, a study conducted by Healy and Hoyles (2000), in the context of algebra 133 teaching, suggests that students prefer arguments that simultaneously convince and justify the 134 relationship in question. A conclusion suggesting that explanation is something important for students 135 and that it can even be a worthy resource for greater use and exploration in the teaching of Mathematics. 136 Interestingly, the situation seems to be interpreted a little differently by some teachers. Indeed, as 137 mentioned by Biza, Nardi and Zachariades (2010), while all teachers recognize the verifying role of 138 proof, the same does not happen in relation to its role in terms of comprehension. Actually, as the 139 authors refer, some teachers tend to check the validity of a mathematical relationship based on 140 examples, even when they have just proved it. Besides that, teachers consider that arguments based on 141 concrete cases or on visual representations have greatest potential to convince.

142 But there are other roles that are also assigned to proof. Bleiler-Baxter and Pair (2017), and several other 143 authors, refer to proof as a discovery process (a function of proof introduced by De Villiers, 2020, 1990). 144 According to them, there are numerous examples in the history of Mathematics of new results that were 145 discovered or invented by purely deductive processes; in fact, it is completely unlikely that some results 146 (such as, for example, non-Euclidean geometries) could ever have been found by mere intuition. The 147 role of proof as a systematization process is also addressed, considering that it reveals the underlying 148 logical relationships between statements in a way that pure intuition would not be able to accomplish. 149 In turn, Davis and Hersh (1983) see proof as an intellectual challenge, considering that it fulfills a 150 gratifying and self-fulfilling function. Proof is therefore a testing ground for intellectual energy and 151 mathematical ingenuity.

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# 153 Knowledge for Teaching Mathematics with Technology – the KTMT model 154 The main goal behind the conception of the KTMT model is the articulation of the research about the 155 teachers' technology integration and the research about the teachers' professional knowledge. The 156 model recognizes the contribution of the work of authors such as Shulman (1986), and Mishra and 157 Koehler (2006) on the definition of the knowledge domains considered and assumes three types of 158 knowledge domains: base knowledge, inter-domains knowledge, integrated knowledge.

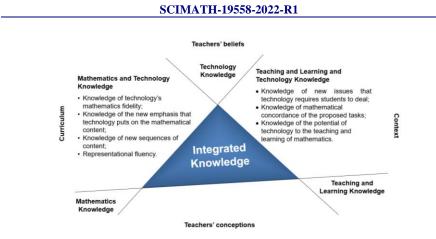
The base knowledge domains are four: Mathematics, Teaching and Learning, Technology, and Curriculum and Context. Curriculum and Context is assumed as a transversal domain, influent on all the other domains. This is a domain that includes all the influences over the teachers' options, being these personal influences (such as the teachers' beliefs) or external influences (such as the school context).

164 Inter-domain knowledge is a type of knowledge central in this model and the main characteristic of it, 165 as well as the main difference from other knowledge models. This type of knowledge is a new 166 knowledge developed from more than one base knowledge and integrating in its characterization 167 results from the research on technology integration. The KTMT model considers two inter-domain 168 knowledge: the Mathematics and Technology Knowledge (MTK), and the Teaching and Learning and 169 Technology Knowledge (TLTK) (figure 1). MTK focuses on how technology influences mathematics, 170 enhancing or constraining certain aspects, and TLTK focuses on how technology affects the teaching and learning process, enhancing or constraining certain approaches. 171

172 Integrated Knowledge (IK) is the last type of knowledge in the KTMT model, developed from the 173 articulation between all the knowledge domains. As the previous mentioned domains of knowledge, 174 this is a new knowledge. It develops from the knowledge held by the teachers in the base domains and 175 in the inter-domains, however, this development does not prevent the continuous development of the 176 knowledge in all the domains. This is an on-going process. The knowledge in all the domains 177 continuous to evolve, generating new knowledge and contributing to the professional development of 178 the teacher.

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179

180

#### Figure 1. KTMT model by Rocha (2020)

181 Integrating knowledge from different domains, such as Mathematics, Teaching and Learning and 182 Technology is assumed as central in the KTMT model. An option also present in other models, such as 183 the TPACK from Mishra and Koehler (2006). However, the way how this integration is conceived is 184 different. And this is a very important characteristic of KTMT and the main difference of this model in 185 comparison to others. MTK and TLTK are not conceived as knowledge resulting from an intersection 186 of knowledge in the base domains. They are new knowledge. A new knowledge resulting from an 187 articulation between two of the base knowledge domains. And this is a dynamic knowledge, a 188 knowledge that continues to be developed, as knowledge in two of the base domains continues to 189 interact and to generate some new knowledge.

The research conducted so far on technology integration has offered some very relevant results. KTMT intends to integrate these results on the model. For instance, the research on technology integration documents students' difficulties, and the KTMT model includes the teachers' awareness of the difficulties faced by the students when using technology as part of the teachers TLTK. There are also studies addressing how technology can impact the mathematics content being addressed, and the model includes knowledge about the new emphasis technology can put on the mathematical content as part of MTK.

- 197 TLTK and MTK are the inter-domain knowledge, and they have a central role in the model. As so, they198 will have a central role in this study.
- 199

200 Methodology

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201 The investigation presented here adopts a qualitative and interpretive approach (Yin, 2017) and focus 202 on the teacher called Teresa. Data collection involved interviews, observing a 10th grade class while 203 studying functions and collecting documents, Semi-structured interviews were carried out before and 204 after each class observed, with the intention of knowing what the teacher had prepared and the reasons 205 for these options (pre-class interviews) and her reflections of the way the class took place (post-class interviews). Both the interviews and the classes were audio-recorded. A logbook of the observed classes 206 207 was also prepared and documents such as worksheets and other materials made available by the teacher to the students were collected. Data analysis was essentially descriptive and interpretive. 208

209 Data analysis was based on the criteria presented in table 1. These criteria were developed from the 210 KTMT model attending to the characteristics of the present study, namely the focus on proof. These 211 criteria were then used to interpret the options assumed by the teacher. First the teacher practice in the 212 classroom was divided in parts (such as launching the task, providing information, supporting the 213 students) and each part was analyzed intending to identify evidence of the defined criteria.

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Knowledge of how technology enables the discovery of mathematical relationships and regularities and of the technology impact on it Knowledge of how technology, by allowing the observation of many cases, can affect the relevance of proof, reducing or even eliminating it	Knowledge of the teaching and learning and of the technology impact on them	Knowledge of the characteristics and potential of exploratory tasks in the context of technology integration Knowledge of students' difficulties in the context of technology integration
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216

217 study taught the topic Functions in Mathematics to a 10th grade class at a school in Portugal and who

218 has a long experience of using graphing calculators with students (the technology used in the study

219 and owned by each of the students) and a deep knowledge of the machine's operation.

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220		
221		C
222	In this section we present one of the tasks (see annex) proposed by the teacher and where in addition	be ac cl
223		fi
224	conjecture (1-teacher, 5-Student, K-Kesearcher).	Core re su
225	Teresa starts the lesson informing the students that they are going to carry out an exploration task and	
226	that this work will be carried out in pairs. She emphasizes this last aspect, stressing the importance of	
227	the collaborative work. This approach gives evidence of the teacher's awareness of the characteristics	
228	of this type of work, also suggesting knowledge about the need to share with the students some of these	
229	characteristics (TLTK).	
230	She then gives some information regarding the operation of the calculator, focusing on what she	
231	considers that the students will need during the task. The technical knowledge of the technology is	
232	shared in this way with the students (TK). Then, she shares her expectations, speaking about which	
233	questions she considers will be easy, which ones could be more difficult and how far she wants	
234	everyone to go. An action showing knowledge about this type of tasks, but also about the students and	
235	the easy way how they can lose notion of time (TLTK):	
236	T - The aim of each pair is to do everything up to question 6. Up to question 5 I think it's easy.	
237	You must do well, as quickly as you can. Question 6 will not be so easy, () here it is expected	
238	that you prove. I think the proof is not very difficult and therefore I have some hope that many	
239	of you will be able to do the proof. The "Going further", which comes in questions 7 and 8, I also	
240	hope that some of you manage to do it. If some of you manage to do these questions, it's very	
241	good because I don't hope that you have time to do it here in class, but I hope that you do it at	
242	home, afterwards. So, the goal is for everyone to do everything up to question 6, including the	
243	proof, for some the goal is to do also question 7 and then, who knows (lesson)	
244	Before encouraging students to start working, the teacher also addresses the issue of proof and its	
245	relevance in Mathematics, briefly discussing central ideas in Mathematics (MK), but also connecting	
246	them with the impact of using technology (MTK). In this approach, Teresa emphasizes to the students	
247	the need to some kind of confirmation before assuming the veracity of a conjecture (TLTK):	
248	T - The sixth question () is a proof and I would like to talk a little bit about it. () In	
249	Mathematics we often experiment. We've already done that here with functions. We have	
250	studied families of functions and then or I give you some information, saying that the conjecture	

**Commented [L7]:** In the results section, it would be better if the coding process of the criteria analyzed is added, so that the direction of the research objectives is clear and the reader can immediately identify the findings.

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251	you formulated is true in all cases, and you believe me, you can also consult the textbook and	
252	etc., or we prove the result is always true. We do what mathematicians always do. In	
253	Mathematics, proof is the essence of the discipline, so we cannot forget about it. (lesson)	
254	From this moment on, the entire lesson takes place centered in the students' work, with the teacher	
255	circulating among the groups and responding to their requests.	
256	When the first conjectures emerge, Teresa feels the need to draw the students' attention to the small	
257	number of examples that were considered in their formulation, but they do not seem very sensitive to	
258	her comments and only the doubt about the veracity of the conjecture seems to led the students to	
259	consider analyzing a few more cases:	
260	T - Are you formulating a conjecture based on just two examples?	
261	S - Oh, but we've already seen it.	<b>Commented [L9]:</b> It will better display the results of student work, thus strengthening the triangulation
262	T - And what did you notice?	process
263	S - It corresponds to multiplication, but it has to be less this times this. () It has to be $-(5 \times 3)$ .	
264	T - Okay, great. It's your guess.	
265	S - () But that's -15. It's wrong. That's why in the next question they ask for an answer if the	
266	points are in the same side of the axis. Isn't it?	
267	T - I don't know. () You only experimented with two examples. You are taking conclusions	
268	based only in two examples you can see more examples, if you have doubts. That way you can	
269	check if you are getting is right or not.	
270	S - How many pairs should we do?	
271	T - In an investigation there is no limit. Do several, until you can reach a conclusion two is very	
272	little to do. I think, don't you? (lesson)	
273	Seeing the quantity of cases analyzed to develop the conjectures, the teacher tries to let the students	
274	think about the confidence they can have in the result formulated. But seeing they are not sensitive to	
275	that, and knowing the importance of letting them explore, she chooses to instill the doubt in their mind	
276	(TLTK).	
277	Not all the students react this way. Some consider that the more examples they do, the better. But even	
278	so, they seem to feel some discomfort for not being given a specific number. And once again, the	

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279 280	knowledge of the teacher guides her action (TLTK) and makes her avoid giving a direct answer and leave the decisions to the students:	
281	S - How many [examples] should we do?	
282	T - That's up to you.	
283	S - As many as we wish. The more the better (lesson)	
284	But in some cases, in addition to the number of examples considered, the conjecture seems to be	
285	formulated in a somewhat thoughtless way, leading Teresa to question the students so that they feel	
286	the need to better ponder the conclusion they reached. Once again, the teacher poses questions, instead	
287	of giving answers, leaving the exploratory work to the students (TLTK):	
288	S - I have already concluded something. The ordinate at the origin is always x1×x2 and then the	
289	slope of the segment is the difference between one and the other.	
290	T - $x1 \times x2$ ? So how much is it 3 × (-5)?	
291	S - No.	<b>Commented [L10]:</b> It will better display the results of
292	T - Tell me, how much is it?	student work, thus strengthening the triangulation process
293	S15.	
294	T15, and there it is?	
295	S - 15.	
296	T - 3× (-4)?	
297	S - It's -12. So okay, it's the other way around.	
298	T - The reverse?	
299	S - Yes.	
300	T- Is it the reverse?	
301	S - Yes. Is it the module? It could be less. The ordinate at the origin is less or	
302	T – So, think about it but write the conclusions. (lesson)	

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303	The proof was the final phase of the wark carried out in the lessen by the students as predicted by	
303 304	The proof was the final phase of the work carried out in the lesson by the students, as predicted by	
304	Teresa, once none of them managed to go beyond this in the available time.	
305	This was a phase of the work in which difficulties arose, something that Teresa already anticipated	
306	(based on her TLTK) and which, as it happened, she intended to address individually, supporting the	
307	students as the problems arose:	
308	T - The proof, even in the simplest case, is still not simple for these 10th grade kids. I will have to	
309	give some tips on the spot and there will be some that do it and there will be others that take a	
310	long time. (pre-lesson interview)	
311	While addressing the question related to proof, however, other issues arise. The first one concerns the	
312	meaning of the term conjecture, with different students questioning its meaning, even after having	
313	already elaborated their conjecture:	
314	S1- Teacher, what is the conjecture?	
514	S1 <sup>-</sup> reacher, what is the conjecture:	
315	T - The conjecture is exactly that. That's what I think will be true. Afterwards, I must prove it. I	
316	think it's true, but I need to prove it really is. While studying Geometry we did that. Here, in the	
317	cases you have seen, it is true (referring to the examples considered by the students) and this	
318	allows me to conjecture, it allows me to think that it will always be true. It's only when I prove	
319	that I'm sure it's always like that. It is true in all the cases.	
320	()	
520	()	
321	T - What is the conjecture? What do you want to conjecture?	
322	S2- But what are we supposed to say by conjecture? (lesson)	
323	But understanding the meaning of the term proof seems to be even more complex. Indeed, some	
324	students seem not to feel the need for generic analytical work, when the cases they analyzed leave them	
325	convinced of the truth of their conjecture:	
326	S - And here in question 6, if we have already shown the calculations here (points to the examples	
327	recorded above) Can I say that this proves the validity of our conjecture?	
328	T – Does it?	
329	S - No? (lesson)	

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330	In fact, instead of trying to prove their conjecture, what many students did was to perform analytically
331	the calculations for the slope and the ordinate at the origin of the cases they had considered graphically.
332	Even so, they have doubts if this is really what is intended:
333	S - We are not understanding question 6.
334	T - It's the proof.
335	S - Do we do the math? Should we put the calculations?
336	T - Right. But you did it for these three cases. Now, for a proof (the student interrupts her)
337	S - Ah! We must do more!
338	T - A proof I mean, to be proved I have to do it for how many cases?
339	S - For many.
340	T - How many? How many?
341	S - Infinite.
342	T - Infinite. (interrupts to ask for silence to the class and then helps the students to find a way of
343	representing a point in a generic form)
344	S - It's complicated.
345	T - It's complicated but we don't give up of something just because it's complicated. () The
346	proof must be analytical, and that it's not possible in the calculator You can try to see many
347	cases, but you cannot see infinite cases. (lesson)
348	The teacher is expecting the students' difficulties (TLTK), but she is also prepared for the students view
349	of proof as something unnecessary (MTK). Teresa considers this is a natural approach for the students,
350	as it follows on from what they have been doing:
351	T - I saw, I don't know how many students now I'm going to see what they wrote, but there
352	were some students that in the proof what did they do? They move to an analytical approach.
353	They approach the same examples, but now using analytical calculations instead of using the
354	calculator. () And this basically corresponds to what we have done in other situations. We
355	don't call it a proof, of course, but it corresponds to work we have done. I have been concerned

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357a transposition of these situations that we have been doing here for this. (post lesson interview)358The articulation between the graphic and the analytic is, therefore, something that Teresa says she pays attention to and that she addresses in the challenges she poses to the students at the end of this task and which she intends to explore in another lesson. Indeed, these last questions come precisely to emphasize the relevance of this choice between the graphic and the analytic approaches. The teacher is just calculation without much usefulness (TLTK). In this case, however, the analytic approach offers the simplest and quickest approach to the question, although not necessarily an easy one (MTK). And the teacher wants her students to be aware of that:366R - In "Going further" the parable becomes another. Do you think it's easy to experiment some cases with the calculator and discover the relationship?368T - No, I don't think so.369R - It's just that I didn't make it. I found it, but I found it analytically. It's also true that I got tired. I gave up and decided to do it analytically.371T - Exactly. But the intention is also that. It's for them to realize that there are things where we don't need to go into calculus, but there are others where calculus is useful. And this calculation is still difficult for them, isn't it? But I prefer to work the calculus like this, so that they realize that there is some advantage in doing some calculus (pre-lesson interview)375The notion that, in order to prove, it is necessary to consider all the cases and not just a few (MTK) is and the students aware of the relevance of proof even when the technology already convinced me about the veracity of my conjecture (MTK), starting from the students' conceptions that she is anticipat	356	about working in the calculator and working analytically and therefore I think they have made
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And that means consider an cases and, in this case, they were minine. (post ressoli interview)	386	And that means consider all cases and, in this case, they were infinite. (post lesson interview)

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In this sense, she even expresses her intention not to close the issue yet. Discussing with the students the proof in the simplest case and leaving the challenges open, to be presented later to the class by some of the students who can solve them. And the teacher makes considerations about the right moment to do it (TLTK), referring to a moment when the calculations needed to prove are being a focus of the lessons (MTK):

- 392 T - I'll do the proof in this case, just for f(x)=x<sup>2</sup>, and I will leave the challenges of "Going further" 393 still open. As they manage to address the challenges, they can write what they did and give it to 394 me. (...) Doing it requires some algebraic manipulation of expressions and they have never 395 worked on it because in the previous school years we don't do this kind of work up to this level. 396 As we are now starting to study the polynomials... The idea is to make them aware of the 397 relevance of these algebraic manipulations, instead of addressing it disconnected from any 398 relevance. So, later, I intend to go back to this, when some of them have already done it. I'll ask 399 one of them to make a presentation to the class, when we are working on calculations with 400 polynomials. (post lesson interview)
- 401 After trying to make students realize that proving requires that all cases are considered and not just a 402 few, Teresa chooses to help students to consider generic points that allow them to effectively prove 403 what is intended. She supports the students work in what she knows they already can do (TLTK) and 404 tries to make them going forward, supporting them in finding a suitable representation and connecting 405 it with their mathematical knowledge and what they experienced with technology (MTK), inspiring 406 them to move from the particular cases to the general one:
- 407T So in question 6 what I'm asking is this: for these points this is true, so now following this408reasoning, if the point are not these... You have two points, then what if it is a point 1, for409example, of coordinates (x1, y1) and a point 2 of coordinates (x2, y2). Now this y1 and this y2 are410not just any ones. Why? These points also belong to the parable. And so, what is it, what is y1?411And y2? (helps the student to get to the answer) So this point is (x1, x1²) and this point is (x2, x2²). (...) Will you now be able to prove? Now prove... you must use what you know. You know413how to calculate the slope of a straight line passing by two points, right? So, let's try to do it.
- 414 S But here, up here we had already shown this.
- 415T You showed, but that's just for one specific case. If you show for this case... you have to do416exactly the same reasoning, but the calculations are a little more complex, you have to do it slowly417and carefully... If you do the same reasoning but for any point, you don't show it for one single418case, you show it for how many cases?

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419	S - To infinite.

Conclusion

420

(...)

T - So if you can do exactly the same reasoning but for this general case... (lesson)

421 It is possible to see that during all the task, the teacher is balancing her approach guided by her TLTK 422 and her MTK. In one hand the teacher is supporting her options in what she knows about this type of 423 tasks and about the students' approaches and difficulties and, in the other hand, she is being guided by 424 the mathematical knowledge she wants to promote, keeping in mind the potential of the technology. 425 This suggests the teacher is guiding her practice by her IK.

426

**Commented [L11]:** Before concluding, it would be better to add a discussion because this section is very important to compare with previous findings

427 428

#### 429 The teachers' MTK influence in the options related to mathematical proof while implementing 430 explorations in a context of technology integration

The teacher's MTK guides her options, leading her to focus on helping students understand what a conjecture is (where the need to ensure its validity deserves emphasis), and what a proof is. The main focus seems to be on this understanding rather than on the proof itself. Still, there is the intention to help students adopt a more formal language, important for the realization of a proof. This domain of knowledge is also responsible for her intention to help students understand the importance of algebraic manipulations, making them feel that it is not just calculations and procedures that they have to learn, but that there is a use for them.

The way how proof is integrated in the task, after a stage of exploration and conjecture formulation,
and with a focus on ensuring the validity of the conjectures, ascribes to the proof the role of verification.
Roles such as the one of understanding are not considered by the teacher in exploration tasks. However,
this option can be more a result of the type of task than of the teacher's MTK.

442

## The impact of the teachers' TLTK in mathematical proof while implementing explorations in a contextof technology integration

Although there is clearly a focus on Mathematics and a set of learnings focused on Mathematics, the
teacher's choices seem essentially guided by her TLTK. And this is because it is the teacher's knowledge
of the students and their difficulties that seems to guide all options. It is the teacher's knowledge of the
type of task and the way in which the students approach them (often advancing and establishing

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449 conclusions based on a very small number of observations) that leads her to reinforce the importance 450 of validating the conjectures. But this is an option that is based on the knowledge of the students, but 451 also on what is the essence of Mathematics. Thus, although the teacher's TLTK is the starting point that 452 guides her practice, an IK is actually present. It is also the knowledge that the teacher has of the students 453 that lead her to define the understanding of the need for proof as fundamental and to recognize that 454 this is still a complex process for students and that it must be progressively developed. But the 455 importance of insisting on this aspect comes from her MTK and so, once again, it is possible to identify 456 an IK. The knowledge about the students' preference for graphical over analytical approaches is also 457 part of the teacher's TLTK. But the teacher's MTK allows her to be aware of the importance of both 458 approaches and, in conjunction with her TLTK (and therefore IK) leads her to deliberately look for 459 opportunities to confront students with situations where both approaches prove useful.

460

#### 461 Final comments

462 The knowledge about the relevance of proof in Mathematics, together with the need to understand 463 what a conjecture is and the difference from a proof; as well as the knowledge about the students and 464 their difficulties, are part of the teacher's MTK and TLTK and guide the teacher's action. The integration 465 made by the teacher between TLTK and MTK (i.e., IK) seems to be of great importance, as it allows the 466 characteristics of an exploratory work not to be abandoned, having the students effectively 467 experimenting and seeking for regularities (TLTK), but, at the same time, it allows to approach the 468 essential characteristics of the Mathematics, namely the need to guarantee the veracity of the 469 conjectures formulated in all cases and not only in those observed (MTK). It seems, therefore, that it is 470 the articulation between the two domains of knowledge at IK that allows for a balance that enhances 471 student learning.

The study provides evidence about the difficulty of articulating proof and technology, in line with the
difficulties addressed in the literature and related to mathematical proof (De Villiers, 1999; Hsieh et al.,
2012), but it also offers evidence of the relevance of this articulation and of how the teacher's
professional knowledge can impact the teacher's options.

- 476
- 477 Acknowledgements
- 479

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480	Notes: <sup>1</sup> Here we assume as an exploration task, a task where the students analyze different situations,
481	trying to infer some regularity, to develop a conjecture.
482	
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550	Annex
551	On the parabola's axis
552	Consider the quadratic function defined by $f(x) = x^2$ .
553 554	<b>1.</b> Represent it graphically in the window: $x \in [-10, 10]$ and $y \in [-8, 30]$ . (-5, 25) (-5, 25) (-5, 25) (0, 15) (3, 9)
555	2. Choose two points on the parabola, one on each side
556	of the vertical axis. For example, points $x_1$ and $x_2$ of
557	abscissas 3 and -5. $\boxed{\bigcirc \blacksquare f2(x)=x^2} \Leftrightarrow$
558	Draw the line joining these two points.
559	Record the ordinate at the origin and the slope of this line.
560	Note Ti-nspire: b 7: Points and lines (Point in an object; line, intersection point)
561	b 1: Actions, 7: Coordinates and equations
562	b 8: Measure, 3: Slope
563	3. Repeat the process for other pairs of points with abscissas of your choice and fill in this table:         Abscissa of X1       3         Abscissa of X2       -5         Slope of the segment       0         Ordered at origin       0
564	<b>4.</b> Make a conjecture about the relationship between the slope of the segment and the abscissas of $x_1$
565	and x <sub>2</sub> .
566	5. Make a conjecture about the relationship between the ordinate at the origin and the abscissas of $x_1$
567	and x2.
568	Will the conjectures be valid if the two points are on the same side of the axis? Confirm.
569	6. Demonstrate the validity of your conjectures.
570	
571	Going further
572	7. What would happen with the function $f(x) = 2x^2 + 5x + 6$ ?

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- 573 Going even further
- 574 8. And in the general case of the function  $f(x) = ax^2 + bx + c$ ?
- 575

1	The impact of teachers' knowledge on the connection between
2	technology supported exploration and mathematical proof
3	ABSTRACT
4	Technology is recognized for its potential to carry out work of an investigative or explorato

ory ogy 5 nature. The ease and speed with which it becomes possible to observe many cases of a given 6 situation, allows the development of conjectures and brings conviction about their veracity. 7 Mathematical proof, assumed as the essence of Mathematics, thus tends to appear to the students as something dispensable. Based on KTMT - Knowledge for Teaching Mathematics 8 9 with Technology model, this study intends to understand the impact of the teachers' knowledge 10 on mathematical proof in a context of technology integration. The study adopts a qualitative 11 and interpretative methodology analyzing the practice of one teacher. The main conclusions 12 emphasize the relevance of the teacher's MTK - Mathematics and Technology Knowledge, to 13 discuss with the students the conditions to consider when formulating a conjecture and the role 14 of proof; and also the relevance of the teacher's TLTK - Teaching and Learning and 15 Technology Knowledge, to anticipate the students difficulties and support them. The study provides evidence about the difficulty of articulating proof and technology, but it also offers 16 17 evidence of the relevance of this articulation and of how the teacher's professional knowledge can impact the teacher's decisions. 18

19 Keywords: professional knowledge, KTMT, technology, proof

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22	
23	Dear Editor,
24	First of all, I would like to thank you and the reviewer for all the attention you gave to my manuscript
25	and for all the comments to improve it.
26	In the next lines I comment on the changes done, marked in the manuscript with different colors as
27 28	follows: Reviewer 1, Reviewer 2, Reviewer 3, More than one reviewer.
20 29	- Title: The title of the manuscript was changed in order to better reflect the focus of
30	the study presented. (reviewer 2)
31	- Introduction: The goal of the manuscript was clarified in order to increase the global
32	coherence. (reviewer 1)
33	- Literature review: The definition of proof assumed was clearly stated. The literature
34	review, as a all, was better articulated, including final remarks intending to allow a
35	better understanding about how the literature inform the study. (reviewer 2)
36	- Methodology: Additional information was included (e.g., methodological options,
37	number of lessons observed, mathematical content being addressed, students and
38	teacher background). (all the reviewers)
39	- Results: The results are presented based on a chronological order. Other options
40	were considered, such as organizing the information according to the focus of the
41	research questions. However, the research questions focus on the two inter-domain
42	knowledge: MTK and TLTK. And as it is possible to see by the results presented,
43	these two types of knowledge are often mobilized in very close moments of the
44	same episode. Presenting the data separately for each type of knowledge (or each
45	question) would result in repetitions and difficulties for the reader. Besides this, the
46	acronyms are very easy to identify in the text, making it easy to have a global view
47	of which one is being address. As so, it is also easy to see which research question
48	is being addresses.
49	- Conclusion: The section was enriched, including closer relations to the results
50	sections and to the literature. (more than one reviewer)
51	- Editorial or minor issues referred by the reviewers were also considered.
52	
53	I really hope this new version of the manuscript corresponds to your expectations.
54	Best regards

55

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## The impact of teachers' knowledge on the connection between technology supported exploration and mathematical proof

#### Abstract:

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74 75 76 Technology is recognized for its potential to carry out work of an investigative or exploratory nature. The ease and speed with which it becomes possible to observe many cases of a given situation, allows the development of conjectures and brings conviction about their veracity. Mathematical proof, assumed as the essence of Mathematics, thus tends to appear to the students as something dispensable. Based on KTMT – *Knowledge for Teaching Mathematics with Technology* model, this study intends to understand the impact of the teachers' knowledge on mathematical proof in a context of technology integration. The study adopts a qualitative and interpretative methodology analyzing the practice of one teacher. The main conclusions emphasize the relevance of the teacher's MTK – *Mathematics and Technology Knowledge*, to discuss with the students the conditions to consider when formulating a conjecture and the role of proof; and also the relevance of the teacher's **TLTK** – *Teaching and Learning and Technology Knowledge*, to anticipate the students difficulties and support them. The study provides evidence about the difficulty of articulating proof and technology, but it also offers evidence of the relevance of this articulation and of how the teacher's professional knowledge can impact the teacher's decisions.

Keywords: professional knowledge, KTMT, technology, proof.

#### Introduction

77 Technology is recognized for its potential for teaching and learning mathematics (Tabach & Trgalová, 78 2019). In particular, the possibilities of carrying out work of an investigative or exploratory nature are 79 highly valued. It makes it possible for the teachers to offer to the students the opportunity to experiment 80 with different mathematical relationships, reflecting on them while trying to identify regularities and 81 formulate conjectures. However, this possibility challenges the teachers' professional knowledge 82 (Rocha, 2020b). The ease and speed with which it becomes possible to observe many cases of a given 83 situation, brings conviction about the veracity of the formulated conjecture and fosters a feeling that 84 nothing else is needed to be sure of it (Hsieh et al., 2012; Rocha, 2020b). Mathematical proof, assumed 85 as the essence of Mathematics by several authors (Blanton & Stylianou, 2014; Dawkins & Weber, 2017; 86 Rocha, 2019; Schoenfeld, 2009), thus tends to appear to the students as something dispensable (Hanna, 87 2001).

88 The potential of technology is also related to the ease of access to different representations (Rocha, 89 2020a). And, once again, this potentiality challenges the teachers' knowledge. The accessibility and 90 apparent simplicity of the graphical representation turns the algebraic approach into something that

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**Commented [L1]:** Adjust to the existing methodology in the "Method" section, namely case studies

**Commented [L2]:** The findings in the abstract section should be adjusted to the research questions and the results of answering the research questions

91 can be circumvented and whose need becomes possible to question. The mastery of algebraic 92 calculations, which in an approach without technology was often the only possible option, thus 93 becomes something expendable. It becomes possible to question the interest in learning and teaching 94 certain algebraic manipulations, as well as the level of fluidity and training that should be required 95 from students.

96 Mathematical proof tends to be related to algebraic approaches (although it does not have to be, as 97 stated by Komatsu (2010)) and the use of technology tends to be related to more intuitive and 98 exploratory approaches based often in graphical representation. As so, not much is known about how 99 to articulate these two approaches. In a previous work (Author), we tried to understand how the 100 teachers conceive proof and an algebraic approach in a context of technology integration, and how they 101 try to turn the algebraic approach relevant to the students. Here, our goal is to understand the impact 102 of the teachers' knowledge on mathematical proof in a context of technology integration. However, our 103 focus is not exactly on the proof itself, but more on the understanding about what a proof is (what 104 characterizes it and how it differs from a conjecture). We adopt the KTMT (Knowledge for Teaching 105 Mathematics with Technology) model (Rocha, 2020b), giving a special attention to the MTK 106 (Mathematics and Technology Knowledge) and to the TLTK (Teaching and Learning and Technology 107 Knowledge) - two of the main knowledge domains in the KTMT model, as we will see in the next 108 section. Based on this conceptualization and considering the use of exploration tasks<sup>1</sup> in the study of 109 functions in 10th grade, we intend to answer the following research questions:

- What is the impact of the teachers' TLTK in mathematical proof while implementing explorations
   in a context of technology integration?
- How does the teachers' MTK influences the decisions related to mathematical proof while
   implementing explorations in a context of technology integration?

114A better understanding of the teachers' professional knowledge will offer a deeper understanding115about how mathematical proof and conjectures are addressed in exploration tasks with the use of

- technology. And knowing how TLTK and MTK impact the teachers' practice will be very important to
  promote the teachers' professional development.
- 118

#### 119 Mathematical proof

120 The literature about mathematical proof has devoted attention to several topics, some of them focusing 121 on the students and some others focusing on the teachers. In what concerns teachers, the research has 122 focused on ways of addressing proof in the classroom and on the teachers' knowledge and professional

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development (Stylianides, Bieda & Morselli, 2016; Stylianides, Stylianides & Weber, 2017).
Nevertheless, and besides all the interest in different topics related to proof and its teaching and
learning, not much attention has been given to proof in a context of technology integration.

- 126 The understanding about what a mathematical proof is, has changed over time (Smith, 2006), and is 127 not consensual even among mathematicians (Miyakawa, Fujita & Jones, 2017; Steele & Rogers, 2012). 128 Steele and Rogers (2012, p. 161) assume proof as "a mathematical argument that is general to a class of 129 mathematical ideas and establishes the truth of a mathematical statement based on mathematical facts 130 that are accepted or that have been previously proven". Bleiler-Baxter and Pair (2017, p. 16), inspired 131 by De Villiers's (1990) work, define proof as "logical deduction that is used to verify, explain, 132 systematize, discover, and communicate mathematics". In the classroom context, Stylianides and Ball 133 (2008) refer to it as a mathematical argument that uses mathematical knowledge considered valid by 134 the students and that does not require additional justifications, it adopts reasoning considered valid 135 and already known by the students (or whose understanding is within their reach), and which is 136 adequately communicated in ways already familiar to the students (or whose understanding is within 137 their reach). And this is the understanding of proof assumed in this study.
- 138 The difficulty in getting students to understand the need for and importance of proof in Mathematics 139 is, according to De Villiers (1999), well known to all secondary school teachers. This difficulty is 140 accentuated when technology is involved because, according to Hsieh et al. (2012), the dynamic 141 character usually offered by it allows the carrying out of work of an experimental nature, which 142 enhances the discovery of properties and the formulation of conjectures. Students can easily experiment 143 and analyze various cases, reflecting on important mathematical ideas and, consequently, reaching a 144 higher level of understanding (Goos & Bennison, 2008). Thus, they acquire the possibility to formulate 145 their own questions and to continue formulating hypotheses and testing them, trying to frame the 146 results in the theory they are trying to formulate (Rocha, 2015).
- 147 The way in which the analysis of different cases is made possible, ends up giving students a feeling of 148 confidence regarding the veracity of the conclusions they establish with the support of technology, 149 which is often enhanced by the way students got used to seeing Mathematics validated, i.e., externally, 150 either by the teacher, the textbook or even the parents (Tall et al., 2012). The need to prove the formulated conjecture may thus not be felt. But if inferring a conclusion from reflection on some 151 152 particular cases is an important activity, it is undoubtedly distinct from proving (Cabassut et al., 2012). 153 Emphasizing to the students the need for and importance of proof will then imply the search for its 154 function.

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De Villiers (2012) considers that, traditionally, the justification or convincing about the validity of a conjecture is seen as the main function of proof, and Knuth (2002) considers that this is really the only role that most teachers recognize to it. In recent decades, this narrow view of the role of proof has been criticized by authors such as Reid (2011), who understand that it has also assumed other important roles for mathematicians and that it can also assume a role of great didactic value in the classroom.

160 For Mejía-Ramos (2005), the search for a deeper understanding is what truly moves mathematicians 161 and what leads them to reject the "alleged" proofs carried out by computational means. A point of view 162 also shared by Bleiler-Baxter and Pair (2017). And this, as highlighted by Hanna (2014), despite the fact 163 of understanding being something remaining relatively undefined. This suggests a role of proof as a 164 means and not so much as an end in itself, encompassing both validation and understanding. In the 165 current reality, in which systems with symbolic algebraic calculus and dynamic geometry programs 166 are easily accessible, it is frequent to obtain a validation of the conjecture with a considerable degree of 167 confidence without a proof (De Villiers, 2012). As so, it becomes difficult to justify the need for a proof 168 exclusively with the need for validation.

169 Technologies can convince us of the veracity of the conjecture, but they do not offer us the justification 170 for that veracity (De Villiers, 2012). And this does not seem to be a question exclusive for 171 mathematicians. Indeed, a study conducted by Healy and Hoyles (2000), in the context of algebra 172 teaching, suggests that students prefer arguments that simultaneously convince and justify the 173 relationship in question. A conclusion suggesting that explanation is something important for students 174 and that it can even be a worthy resource for greater use and exploration in the teaching of Mathematics. 175 Interestingly, the situation seems to be interpreted a little differently by some teachers. Indeed, as 176 mentioned by Biza, Nardi and Zachariades (2010), while all teachers recognize the verifying role of 177 proof, the same does not happen in relation to its role in terms of comprehension. Actually, as the 178 authors refer, some teachers tend to check the validity of a mathematical relationship based on 179 examples, even when they have just proved it. Besides that, teachers consider that arguments based on 180 concrete cases or on visual representations have greatest potential to convince.

181But there are other roles that are also assigned to proof. Bleiler-Baxter and Pair (2017), and several other182authors, refer to proof as a discovery process (a function of proof introduced by De Villiers, 2020, 1990).183According to them, there are numerous examples in the history of Mathematics of new results that were184discovered or invented by purely deductive processes; in fact, it is completely unlikely that some results185(such as, for example, non-Euclidean geometries) could ever have been found by mere intuition. The186role of proof as a systematization process is also addressed, considering that it reveals the underlying187logical relationships between statements in a way that pure intuition would not be able to accomplish.

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188	In turn, Davis and Hersh (1983) see proof as an intellectual challenge, considering that it fulfills a
189	gratifying and self-fulfilling function. Proof is therefore a testing ground for intellectual energy and
190	mathematical ingenuity.
191	Thus, the literature highlights the need to better understand the articulation between explorations made
192	with technology and mathematical proof, suggesting difficulties on the part of teachers in this
193	articulation. It also points to different functions of proof, identifying different potentialities, but also
194	showing the existence of different valuations by teachers. And this are issues somehow addressed in
195	this study and closely related to the teachers' professional knowledge.
196	
197	Knowledge for Teaching Mathematics with Technology – the KTMT model
198	The main goal behind the conception of the KTMT model is the articulation of the research about the
199	teachers' technology integration and the research about the teachers' professional knowledge. The
200	model recognizes the contribution of the work of authors such as Shulman (1986), and Mishra and
201	Koehler (2006) on the definition of the knowledge domains considered and assumes three types of
202	knowledge domains: base knowledge, inter-domains knowledge, integrated knowledge.
203	The base knowledge domains are four: Mathematics, Teaching and Learning, Technology, and
204	Curriculum and Context. Curriculum and Context is assumed as a transversal domain, influent on all
205	the other domains. This is a domain that includes all the influences over the teachers' options, being
206	these personal influences (such as the teachers' beliefs) or external influences (such as the school
207	context).
208	Inter-domain knowledge is a type of knowledge central in this model and the main characteristic of it,
209	as well as the main difference from other knowledge models. This type of knowledge is a new
210	knowledge developed from more than one base knowledge and integrating in its characterization
211	results from the research on technology integration. The KTMT model considers two inter-domain
212	knowledge: the Mathematics and Technology Knowledge (MTK), and the Teaching and Learning and
213	Technology Knowledge (TLTK) (figure 1). MTK focuses on how technology influences mathematics,
214	enhancing or constraining certain aspects, and TLTK focuses on how technology affects the teaching
215	and learning process, enhancing or constraining certain approaches.
216	Integrated Knowledge (IK) is the last type of knowledge in the KTMT model, developed from the

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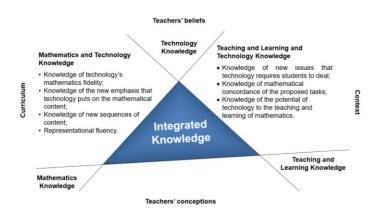
articulation between all the knowledge domains. As the previous mentioned domains of knowledge,

this is a new knowledge. It develops from the knowledge held by the teachers in the base domains and

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219	in the inter-domains, however, this development does not prevent the continuous development of the
220	knowledge in all the domains. This is an on-going process. The knowledge in all the domains
221	continuous to evolve, generating new knowledge and contributing to the professional development of
222	the teacher.



223 224

#### Figure 1. KTMT model by Rocha (2020b)

225 Integrating knowledge from different domains, such as Mathematics, Teaching and Learning and 226 Technology is assumed as central in the KTMT model. An option also present in other models, such as 227 the TPACK from Mishra and Koehler (2006). However, the way how this integration is conceived is 228 different. And this is a very important characteristic of KTMT and the main difference of this model in 229 comparison to others. MTK and TLTK are not conceived as knowledge resulting from an intersection 230 of knowledge in the base domains. They are new knowledge. A new knowledge resulting from an 231 articulation between two of the base knowledge domains. And this is a dynamic knowledge, a 232 knowledge that continues to be developed, as knowledge in two of the base domains continues to 233 interact and to generate some new knowledge.

The research conducted so far on technology integration has offered some very relevant results. KTMT intends to integrate these results on the model. For instance, the research on technology integration documents students' difficulties, and the KTMT model includes the teachers' awareness of the difficulties faced by the students when using technology as part of the teachers TLTK. There are also studies addressing how technology can impact the mathematics content being addressed, and the model includes knowledge about the new emphasis technology can put on the mathematical content as part of MTK.

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TLTK and MTK are the inter-domain knowledge, and they have a central role in the model. As so, theywill have a central role in this study.

243

#### 244 Methodology

The investigation presented here adopts a qualitative and interpretive approach, based on a case study, 245 246 (Yin, 2017) and focus on the teacher called Teresa. Data collection involved interviews, observing a 10th 247 grade class while studying functions and collecting documents. Semi-structured interviews were 248 carried out before and after each class observed, with the intention of knowing what the teacher had 249 prepared and the reasons for these options (pre-class interviews) and her reflections of the way the 250 class took place (post-class interviews). 14 lessons, where the teacher was planning to use technology, 251 were observed while the students were studying functions of several types (linear, quadratic, absolute 252 value, defined by branches). Both the interviews and the classes were audio-recorded. A logbook of the 253 observed classes was also prepared and documents such as worksheets and other materials made 254 available by the teacher to the students were collected. Data analysis was essentially descriptive and 255 interpretive.

Data analysis was based on the criteria presented in table 1. These criteria were developed from the KTMT model attending to the characteristics of the present study, namely the focus on proof. These criteria were then used to interpret the options assumed by the teacher. As a first step, the teacher practice in the classroom was divided in parts (such as launching the task, providing information, supporting the students) and then each part was analyzed intending to identify evidence of the defined criteria.

**Commented [L3]:** In the methodology section, especially the research stages, the first step is written. But there is no description for the next step (second, third and so on)

Table 1. Analysis criteria

	МТК	TLTK	
Knowledge of the Mathematics and of the technology impact on it	Knowledge of how technology enables the discovery of mathematical relationships and regularities Knowledge of how technology, by allowing the observation of many cases, can affect the relevance of proof, reducing or even eliminating it	Knowledge of the teaching and learning and of the technology impact on them	Knowledge of the characteristics and potential of exploratory tasks in the context of technology integration Knowledge of students' difficulties in the context of technology integration
study taught the topic has a long experience	s study is a teacher with over 3 c Functions in Mathematics to e of using graphing calculator f the students) and a deep kn	o a 10th grade class	at a school in Portugal and w
aware of the students	' limited experience as well as	s the difficulties fac	ed when requested to produc
mathematical proof a	nd since the begin of the top	oic previously stud	ied (Geometry) she is trying
familiarize her stude	nts with the characteristics of	of a mathematical	argument and proof. Since
	ol year (about two months be	•	2,
are also becoming far	niliar with exploration tasks	· · ·	, ,
		1 111 116	
, , , , , , , , , , , , , , , , , , ,	cted to explore several examp	les and identify reg	ularities. The formulation of

### 276 Results

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In this section we present one of the tasks (see annex) proposed by the teacher and where, in addition
to formulating a conjecture regarding a mathematical situation, students are asked to prove their
conjecture (T-teacher, S-Student, R-Researcher).

280 Teresa starts the lesson informing the students that they are going to carry out an exploration task and 281 that this work will be carried out in pairs. She emphasizes this last aspect, stressing the importance of 282 the collaborative work. This approach gives evidence of the teacher's awareness of the characteristics

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262

of this type of work, also suggesting knowledge about the need to share with the students some of these
characteristics (TLTK).
She then gives some information regarding the operation of the calculator, focusing on what she
considers that the students will need during the task. The technical knowledge of the technology is
shared in this way with the students (TK). Then, she shares her expectations, speaking about which
questions she considers will be easy, which ones could be more difficult and how far she wants
everyone to go. An action showing knowledge about this type of tasks, but also about the students and

290 the easy way how they can lose notion of time (TLTK):

- 291 T - The aim of each pair is to do everything up to question 6. Up to question 5 I think it's easy. 292 You must do well, as quickly as you can. Question 6 will not be so easy, (...) here it is expected 293 that you prove. I think the proof is not very difficult and therefore I have some hope that many 294 of you will be able to do the proof. The "Going further", which comes in questions 7 and 8, I also 295 hope that some of you manage to do it. If some of you manage to do these questions, it's very 296 good because I don't hope that you have time to do it here in class, but I hope that you do it at 297 home, afterwards. So, the goal is for everyone to do everything up to question 6, including the 298 proof, for some the goal is to do also question 7 and then, who knows... (lesson)
- Before encouraging students to start working, the teacher also addresses the issue of proof and its relevance in Mathematics, briefly discussing central ideas in Mathematics (MK), but also connecting them with the impact of using technology (MTK). In this approach, Teresa emphasizes to the students the need to some kind of confirmation before assuming the veracity of a conjecture (TLTK):
- 303T The sixth question (...) is a proof and I would like to talk a little bit about it. (...) In304Mathematics we often experiment. We've already done that here with functions. We have305studied families of functions and then or I give you some information, saying that the conjecture306you formulated is true in all cases, and you believe me, you can also consult the textbook and307etc., or we prove the result is always true. We do what mathematicians always do. In308Mathematics, proof is the essence of the discipline, so we cannot forget about it. (lesson)
- From this moment on, the entire lesson takes place centered in the students' work, with the teachercirculating among the groups and responding to their requests.
- 311 The first conjecture of one group of students was based on two examples and states that the line passing 312 through two points of the parabola defined by  $y = x^2$  will cross the y-axis at the symmetric of the
- 313 product of the abscissas of the two points. Being two observations a very small number, Teresa feels

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314	the need to draw the students' attention to <mark>that, trying to call their attention to the risk of establishing</mark>				
315	<mark>conclusions based in</mark> the small number of examples that were considered in their formulation <mark>. But the</mark>				
316	students do not seem very sensitive to her comments and only the doubt about the veracity of the				
317	conjecture seems to lead them to consider analyzing a few more cases:				
318	T - Are you formulating a conjecture based on just two examples?				
319	S - Oh, but we've already seen it.				
320	T - And what did you notice?				
321	S - It corresponds to multiplication, but it has to be less this times this. () It has to be $-(5 \times 3)$ .				
322	T - Okay, great. It's your guess.				
323	S - () But that's -15. It's wrong. That's why in the next question they ask for an answer if the				
324	points are in the same side of the axis. Isn't it?				
325	T - I don't know. () You only experimented with two examples. You are taking conclusions				
326	based only in two examples you can see more examples, if you have doubts. That way you can				
327	check if you are getting it right or not.				
328	S - How many pairs should we do?				
329	T - In an investigation there is no limit. Do several, until you can reach a conclusion two is very				
330	little to do. I think, don't you? (lesson)				
331	Seeing the quantity of cases analyzed to develop the conjectures, the teacher tries to let the students				
332	think about the confidence they can have in the result formulated. But seeing they are not sensitive to				
333	that, and knowing the importance of letting them explore, she chooses to instill the doubt in their mind				
334	(TLTK).				
335	Not all the students react this way. Some consider that the more examples they do, the better. But even				
336	so, they seem to feel some discomfort for not being given a specific number. And once again, the				
337	knowledge of the teacher guides her action (TLTK) and makes her avoid giving a direct answer and				
338	leave the decisions to the students:				
339	S - How many [examples] should we do?				
340	T - That's up to you.				

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341	S - As many as we wish. The more the better (lesson)
342	But in some cases, in addition to the number of examples considered, the conjecture seems to be
343	formulated in a somewhat thoughtless way, leading Teresa to question the students so that they feel
344	the need to better ponder the conclusion they reached. Once again, the teacher poses questions, instead
345	of giving answers, leaving the exploratory work to the students (TLTK):
346	S - I have already concluded something. The ordinate at the origin is always $x_1 \times x_2$ and then the
347	slope of the segment is the difference between one and the other.
348	T - $x_1 \times x_2$ ? So how much is it 3 × (-5)?
349	S - No.
350	T - Tell me, how much is it?
351	S15.
352	T - –15, and there it is?
353	S - 15.
354	T - 3× (-4)?
355	S - It's -12. So okay, it's the other way around, it's the reverse.
356	T - The reverse?
357	S - Yes.
358	T- Is it the reverse?
359	S - Yes. Is it the module? It could be less. The ordinate at the origin is less or
360	T – So, think about it but write the conclusions. (lesson)
361	The proof was the final phase of the work carried out in the lesson by the students, as predicted by
362	Teresa, once none of them managed to go beyond this in the available time.
363	This was a phase of the work in which difficulties arose, something that Teresa already anticipated
364	(based on her TLTK) and which, as it happened, she intended to address individually, supporting the
365	students as the problems arose:

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366 367	T - The proof, even in the simplest case, is still not simple for these 10th grade kids. I will have to give some tips on the spot and there will be some that do it and there will be others that take a long time (pre-lesson interview).				
368	long time. (pre-lesson interview)				
369	While addressing the question related to proof, however, other issues arise. The first one concerns the				
370	meaning of the term conjecture, with different students questioning its meaning, even after having				
371	already elaborated their conjecture:				
372	S1- Teacher, what is the conjecture?				
373	T - The conjecture is exactly that. That's what I think will be true. Afterwards, I must prove it. I				
374	think it's true, but I need to prove it really is. While studying Geometry we did that. Here, in the				
375	cases you have seen, it is true (referring to the examples considered by the students) and this				
376	allows me to conjecture, it allows me to think that it will always be true. It's only when I prove				
377	that I'm sure it's always like that. It is true in all the cases.				
378	()				
379	T - What is the conjecture? What do you want to conjecture?				
380	S2- But what are we supposed to say by conjecture? (lesson)				
380 381	S2- But what are we supposed to say by conjecture? (lesson) But understanding the meaning of the term proof seems to be even more complex. Indeed, some				
381	But understanding the meaning of the term proof seems to be even more complex. Indeed, some				
381 382	But understanding the meaning of the term proof seems to be even more complex. Indeed, some students seem not to feel the need for generic analytical work, when the cases they analyzed leave them				
381 382 383	But understanding the meaning of the term proof seems to be even more complex. Indeed, some students seem not to feel the need for generic analytical work, when the cases they analyzed leave them convinced of the truth of their conjecture:				
381 382 383 384	But understanding the meaning of the term proof seems to be even more complex. Indeed, some students seem not to feel the need for generic analytical work, when the cases they analyzed leave them convinced of the truth of their conjecture: S - And here in question 6, if we have already shown the calculations here (points to the examples				
381 382 383 384 385	But understanding the meaning of the term proof seems to be even more complex. Indeed, some students seem not to feel the need for generic analytical work, when the cases they analyzed leave them convinced of the truth of their conjecture: S - And here in question 6, if we have already shown the calculations here (points to the examples recorded above) Can I say that this proves the validity of our conjecture?				
<ul> <li>381</li> <li>382</li> <li>383</li> <li>384</li> <li>385</li> <li>386</li> </ul>	But understanding the meaning of the term proof seems to be even more complex. Indeed, some students seem not to feel the need for generic analytical work, when the cases they analyzed leave them convinced of the truth of their conjecture: S - And here in question 6, if we have already shown the calculations here (points to the examples recorded above) Can I say that this proves the validity of our conjecture? T – Does it?				
<ul> <li>381</li> <li>382</li> <li>383</li> <li>384</li> <li>385</li> <li>386</li> <li>387</li> </ul>	But understanding the meaning of the term proof seems to be even more complex. Indeed, some students seem not to feel the need for generic analytical work, when the cases they analyzed leave them convinced of the truth of their conjecture: S - And here in question 6, if we have already shown the calculations here (points to the examples recorded above) Can I say that this proves the validity of our conjecture? T – Does it? S - No? (lesson)				
<ul> <li>381</li> <li>382</li> <li>383</li> <li>384</li> <li>385</li> <li>386</li> <li>387</li> <li>388</li> </ul>	But understanding the meaning of the term proof seems to be even more complex. Indeed, some students seem not to feel the need for generic analytical work, when the cases they analyzed leave them convinced of the truth of their conjecture:         S - And here in question 6, if we have already shown the calculations here (points to the examples recorded above) Can I say that this proves the validity of our conjecture?         T - Does it?         S - No? (lesson)         In fact, instead of trying to prove their conjecture, what many students did was to perform analytically				
381 382 383 384 385 386 386 387 388 389	<ul> <li>But understanding the meaning of the term proof seems to be even more complex. Indeed, some students seem not to feel the need for generic analytical work, when the cases they analyzed leave them convinced of the truth of their conjecture:</li> <li>S - And here in question 6, if we have already shown the calculations here (points to the examples recorded above) Can I say that this proves the validity of our conjecture?</li> <li>T - Does it?</li> <li>S - No? (lesson)</li> </ul> In fact, instead of trying to prove their conjecture, what many students did was to perform analytically the calculations for the slope and the ordinate at the origin of the cases they had considered graphically.				

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393	S - Do we do the math? Should we put the calculations?
394	T - Right. But you did it for these three cases. Now, for a proof (the student interrupts her)
395	S - Ah! We must do more!
396	T - A proof I mean, to be proved I have to do it for how many cases?
397	S - For many.
398	T - How many? How many?
399	S - Infinite.
400	T - Infinite. (interrupts to ask for silence to the class and then helps the students to find a way of
401	representing a point in a generic form)
402	S - It's complicated.
403	T - It's complicated but we don't give up of something just because it's complicated. () The
404	proof must be analytical, and that it's not possible in the calculator You can try to see many
405	cases, but you cannot see infinite cases. (lesson)
406	The teacher is expecting the students' difficulties (TLTK), but she is also prepared for the students view
407	of proof as something unnecessary (MTK). Teresa considers this is a natural approach for the students,
408	as it follows on from what they have been doing:
409	T - I saw, I don't know how many students now I'm going to see what they wrote, but there
410	were some students that in the proof what did they do? They move to an analytical approach.
411	They approach the same examples, but now using analytical calculations instead of using the
412	calculator. () And this basically corresponds to what we have done in other situations. We
413	don't call it a proof, of course, but it corresponds to work we have done. I have been concerned
414	about working in the calculator and working analytically and therefore I think they have made
415	a transposition of these situations that we have been doing here for this. (post lesson interview)
416	The articulation between the graphic and the analytic is, therefore, something that Teresa says she pays
417	attention to and that she addresses in the challenges she poses to the students at the end of this task
418	and which she intends to explore in another lesson. Indeed, these last questions come precisely to
419	emphasize the relevance of this choice between the graphic and the analytic approaches. The teacher
420	considers that students generally prefer the graph approach over the analytical, thinking that the latter

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421	is just calculation without much usefulness (TLTK). In this case, however, the analytic approach offers			
422	the simplest and quickest approach to the question, although not necessarily an easy one (MTK). And			
423	the teacher wants her students to be aware of that:			
424	R - In "Going further" the <mark>parabola</mark> becomes another. Do you think it's easy to experiment some			
425	cases with the calculator and discover the relationship?			
426	T - No, I don't think so.			
427	R - It's just that I didn't make it. I found it, but I found it analytically. It's also true that I got tired.			
428	I gave up and decided to do it analytically.			
429	T - Exactly. But the intention is also that. It's for them to realize that there are things where we			
430	don't need to go into calculus, but there are others where calculus is useful. And this calculation			
431	is still difficult for them, isn't it? But I prefer to work the calculus like this, so that they realize			
432	that there is some advantage in doing some calculus (pre-lesson interview)			
433	The notion that, in order to prove, it is necessary to consider all the cases and not just a few (MTK) is			
434	something that she believes needs to be worked on over time (TLTK). In this task her main goal is to			
435	make the students aware of the relevance of proof even when the technology already convinced me			
436	about the veracity of my conjecture (MTK), starting from the students' conceptions that she is			
437	anticipating (TLTK):			
438	T - I expected them to have difficulties in the proof. () The idea is exactly to go on with this			
439	discussion with them then I as I gave them until Wednesday to finish all the questions in the			
440	task, so it will probably be in the Wednesday lesson, I will give back to them what they wrote,			
441	and we will go back to the discussion about the difference between trying one, two, three cases			
442	or doing () And I will discuss with them mainly this question: what does it mean to prove.			
443	The task asks them to include the examples they've already done, but it also asks them to prove.			
444	And that means consider all cases and, in this case, they were infinite. (post lesson interview)			
445	In this sense, she even expresses her intention not to close the issue yet. Discussing with the students			
446	the proof in the simplest case and leaving the challenges open, to be presented later to the class by some			
447	of the students who can solve them. And the teacher makes considerations about the right moment to			
448	do it (TLTK), referring to a moment when the calculations needed to prove are being a focus of the			
449	lessons (MTK):			

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450	T - I'll do the proof in this case, just for $f(x)=x^2$ , and I will leave the challenges of "Going further"
451	still open. As they manage to address the challenges, they can write what they did and give it to
452	me. () Doing it requires some algebraic manipulation of expressions and they have never
453	worked on it because in the previous school years we don't do this kind of work up to this level.
454	As we are now starting to study the polynomials The idea is to make them aware of the
455	relevance of these algebraic manipulations, instead of addressing it disconnected from any
456	relevance. So, later, I intend to go back to this, when some of them have already done it. I'll ask
457	one of them to make a presentation to the class, when we are working on calculations with
458	polynomials. (post lesson interview)

After trying to make students realize that proving requires that all cases are considered and not just a few, Teresa chooses to help students to consider generic points that allow them to effectively prove what is intended. She supports the students work in what she knows they already can do (TLTK) and tries to make them going forward, supporting them in finding a suitable representation and connecting it with their mathematical knowledge and what they experienced with technology (MTK), inspiring them to move from the particular cases to the general one:

- 465 T So in question 6 what I'm asking is this: for these points this is true, so now following this 466 reasoning, if the point are not these... You have two points, then what if it is a point 1, for 467 example, of coordinates  $(x_1, y_1)$  and a point 2 of coordinates  $(x_2, y_2)$ . Now this  $y_1$  and this  $y_2$  are 468 not just any ones. Why? These points also belong to the **parabola**. And so, what is it, what is  $y_1$ ? 469 And  $y_2$ ? (helps the student to get to the answer) So this point is  $(x_1, x_1^2)$  and this point is  $(x_2, x_2^2)$ . 470 (...) Will you now be able to prove? Now prove... you must use what you know. You know how 471 to calculate the slope of a straight line passing by two points, right? So, let's try to do it.
- 472 S But here, up here we had already shown this.

T - You showed, but that's just for one specific case. If you show for this case... you have to do
exactly the same reasoning, but the calculations are a little more complex, you have to do it slowly
and carefully... If you do the same reasoning but for any point, you don't show it for one single
case, you show it for how many cases?

477 S - To infinite. (...)

478 T - So if you can do exactly the same reasoning but for this general case... (lesson)

479 It is possible to see that during all the task, the teacher is balancing her approach guided by her TLTK480 and her MTK. In one hand the teacher is supporting her options in what she knows about this type of

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481	tasks and about the students' approaches and difficulties and, in the other hand, she is being guided by	
482	the mathematical knowledge she wants to promote, keeping in mind the potential of the technology.	
483	This suggests the teacher is guiding her practice by her IK.	
484		
485	Conclusion	Com
486	The main goal of this study is to understand the impact of the teachers' knowledge on mathematical	lebih c hasil te Selanji
487	proof in a context of technology integration, giving a special attention to the impact of the teacher's	jawaba
488	MTK and TLTK.	
489 490	The teachers' MTK influence in the decisions related to mathematical proof while implementing explorations in a context of technology integration	
491	The teacher's MTK guides her decisions, leading her to focus on helping students understand what a	
492	conjecture is (where the need to ensure its validity deserves emphasis, as addressed by De Villiers,	
493	1999), and what a proof is. The main focus seems to be on this understanding rather than on the proof	
494	itself. Still, there is the intention to help students adopt a more formal language (with all the challenges	
495	included, Aristidou, 2020), important for the realization of a proof (where the teacher tries to help the	
496	students to considerer a general point and not a specific one). This domain of knowledge is also	
497	responsible for her intention to help students understand the importance of algebraic manipulations,	
498	making them feel that it is not just calculations and procedures that they have to learn, but that there is	
499	a use for them (present in the way how the relevance of proof is presented to the students, but also in	
500	the challenges at the end of the task and left to a later moment).	
501	The way how proof is integrated in the task, after a stage of exploration and conjecture formulation,	
502	and with a focus on ensuring the validity of the conjectures, ascribes to the proof the role of verification.	
503	Roles such as the one of understanding are not considered by the teacher in exploration tasks. However,	
504	this option can be more a result of the type of task than of the teacher's MTK. The evidence available	
505	does not allow us to conclude that the teacher is not aware of the different roles of proof addressed in	
506	the literature (De Villiers, 2012) or even that she does not value them (Knuth, 2002).	
507		
508	The impact of the teachers' TLTK in mathematical proof while implementing explorations in a context	
509	of technology integration	
510	Although there is clearly a focus on Mathematics and a set of learnings focused on Mathematics, the	

511 teacher's choices seem essentially guided by her TLTK. And this is because it is the teacher's knowledge

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**Commented [L4]:** Bagian "Kesimpulan", ini mungkin lebih cocok bagian pembahasan karena mengkonfirmasi hasil temuan dengan hasil penelitian-penelitian sebelumnya. Selanjutnya dibuat bagian kesimpulan yang berisikan jawaban dari pertanyaan penelitian

512 of the students and their difficulties that seems to guide all the decisions. It is the teacher's knowledge 513 of the type of task (as suggested by Rocha, 2020b) and the way in which the students approach them 514 (often advancing and establishing conclusions based on a very small number of observations) that leads 515 her to reinforce the importance of validating the conjectures, in line with the work of Hsieh et al. (2012) 516 (trying to make the students understand the relevance of thinking carefully, based on a set of cases, 517 before formulating a conjecture; and transmitting the idea that a conjecture is something that seems to 518 be true, but requiring a deeper analysis -the proof- before it is possible to be sure it is always true). And 519 this is a decision that is based on the knowledge of the students, but also on what is the essence of 520 Mathematics, as assumed by Blanton and Stylianou (2014), Dawkins and Weber (2017), Rocha (2019) 521 and Schoenfeld (2009) (the teacher is aware about how the students can be convinced of the validity of 522 a result based on the observation of some cases; but she also knows the relevance of proof in 523 Mathematics). Thus, although the teacher's TLTK is the starting point that guides her practice, an IK is 524 actually present. It is also the knowledge that the teacher has of the students that leads her to define the 525 understanding of the need for proof as fundamental (when designing the task, the teacher decides to 526 go forward and does not accept to finish the work with the students development of the conjecture) 527 and to recognize that this is still a complex process for the students and that it must be progressively 528 developed (realizing that the students need help to write a general point, and understanding the 529 difference between conjecture and proof as a first step and the proof as a challenge for most of the 530 students). But the importance of insisting on this aspect, an issue addressed by Cabassut et al. (2012), 531 comes from her MTK and so, once again, it is possible to identify an IK. The knowledge about the 532 students' preference for graphical over analytical approaches is also part of the teacher's TLTK (she is 533 expecting that the students do not feel the need to prove, convinced by what they observed with the 534 technology). But the teacher's MTK allows her to be aware of the importance of both approaches and, in conjunction with her TLTK (and therefore IK) leads her to deliberately look for opportunities to 535 536 confront students with situations where both approaches prove useful.

537

### 538 Final comments

The knowledge about the relevance of proof in Mathematics, together with the need to understand what a conjecture is and the difference from a proof; as well as the knowledge about the students and their difficulties, are part of the teacher's MTK and TLTK and guide the teacher's action. The integration made by the teacher between TLTK and MTK (i.e., IK) seems to be of great importance, as it allows the characteristics of an exploratory work not to be abandoned, having the students effectively experimenting and seeking for regularities (TLTK), but, at the same time, it allows to approach the

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545	essential characteristics of the Mathematics, namely the need to guarantee the veracity of the					
546	conjectures formulated in all cases and not only in those observed (MTK). It seems, therefore, that it is					
547	the articulation between the two domains of knowledge at IK that allows for a balance that enhances					
548	student learning.					
549	The study provides evidence about the difficulty of articulating proof and technology, in line with the					
550	difficulties addressed in the literature and related to mathematical proof (De Villiers, 1999; Hsieh et al.,					
551	2012), but it also offers evidence of the relevance of this articulation and of how the teacher's					
552	professional knowledge can impact the teacher's options.					
553						
554	Acknowledgements					
555	???????????????????????????????????????					
556						
557	Notes: <sup>1</sup> Here we assume as an exploration task, a task where the students analyze different situations,					
558	trying to infer some regularity, to develop a conjecture.					
559						
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628				
629	Annex			
630	On the parabola's axis			
	-			
631	Consider the quadratic function defined by $f(x) = x^2$ .			
631 632 633	<b>1.</b> Represent it graphically in the window: $x \in [-10, 10]$ and $y \in [-8, 30]$ .			
632	<b>1.</b> Represent it graphically in the window: $x \in [-10, 10]$ (-5, 25) -2 (0, 15) <b>f2</b> (x)=x <sup>2</sup>			
632 633	<b>1.</b> Represent it graphically in the window: $x \in [-10, 10]$ and $y \in [-8, 30]$ . (-5, 25) (-5, 25) (-			
632 633 634	<ol> <li>Represent it graphically in the window: x∈[-10, 10] and y∈[-8, 30].</li> <li>Choose two points on the parabola, one on each side of the vertical axis. For example, points x<sub>1</sub> and x<sub>2</sub> of <i>I</i> = 1 = 1 = 1 = 1 = 1 = 1 = 1 = 1 = 1 =</li></ol>			
632 633 634 635	<ol> <li>Represent it graphically in the window: x∈[-10, 10] and y∈[-8, 30].</li> <li>Choose two points on the parabola, one on each side of the vertical axis. For example, points x<sub>1</sub> and x<sub>2</sub> of the vertical axis. For example, points x<sub>1</sub> and x<sub>2</sub> of the vertical axis.</li> </ol>			
632 633 634 635 636	<ol> <li>Represent it graphically in the window: x∈[-10, 10] and y∈[-8, 30].</li> <li>Choose two points on the parabola, one on each side of the vertical axis. For example, points x<sub>1</sub> and x<sub>2</sub> of abscissas 3 and -5.</li> </ol>			
632 633 634 635 636 637	<ol> <li>Represent it graphically in the window: x∈[-10, 10] and y∈[-8, 30].</li> <li>Choose two points on the parabola, one on each side of the vertical axis. For example, points x<sub>1</sub> and x<sub>2</sub> of abscissas 3 and -5. Draw the line joining these two points.</li> </ol>			
<ul> <li>632</li> <li>633</li> <li>634</li> <li>635</li> <li>636</li> <li>637</li> <li>638</li> </ul>	<ol> <li>Represent it graphically in the window: xε[-10, 10] and yε[-8, 30].</li> <li>Choose two points on the parabola, one on each side of the vertical axis. For example, points x<sub>1</sub> and x<sub>2</sub> of abscissas 3 and -5. Draw the line joining these two points. Record the ordinate at the origin and the slope of this line.</li> </ol>			
<ul> <li>632</li> <li>633</li> <li>634</li> <li>635</li> <li>636</li> <li>637</li> <li>638</li> <li>639</li> </ul>	<ol> <li>Represent it graphically in the window: xe[-10, 10] and ye[-8, 30].</li> <li>Choose two points on the parabola, one on each side of the vertical axis. For example, points x<sub>1</sub> and x<sub>2</sub> of abscissas 3 and -5. Draw the line joining these two points. Record the ordinate at the origin and the slope of this line. Note Ti-nspire: b 7: Points and lines (Point in an object; line, intersection point)</li> </ol>			

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		Abscissa of $x_1$	3	
		Abscissa of $x_2$	-5	
		Slope of the segment		
		Ordered at origin		
643	4. Mak	e a conjecture about the re	lationship between the slope of the segment and the abscissas of $x_1$	
644	and $x_2$ .			
645	5. Mak	e a conjecture about the re	elationship between the ordinate at the origin and the abscissas of $x_1$	
646	and $x_2$ .			
		-		
647	VV111	the conjectures be valid if	the two points are on the same side of the axis? Confirm.	
648	6. Demonstrate the validity of your conjectures.			
649				
650	Going fi	urther		
651	7. What would happen with the function $f(x) = 2x^2 + 5x + 6$ ?			
652	Going even further			
653	8. And in the general case of the function $f(x) = ax^2 + bx + c^2$			
654				

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